STELLAR SUBDIVISIONS AND STANLEY-REISNER RINGS OF GORENSTEIN COMPLEXES

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Abstract. Unprojection theory aims to analyze and construct complicated commutative rings in terms of simpler ones. Our main result is that, on the algebraic level of Stanley–Reisner rings, stellar subdivisions of non-acyclic Gorenstein simplicial complexes correspond to unprojections of type Kustin–Miller. As an application, we inductively calculate the minimal graded free resolutions of Stanley–Reisner rings associated to stacked polytopes.

1. Introduction

Stanley–Reisner rings of simplicial complexes form an important class of commutative rings whose theory has provided spectacular applications to combinatorics; see [29] and [9, Chapter 5] [21]. The Stanley–Reisner ring of a simplicial complex Δ, defined as the quotient of a polynomial ring by a certain ideal, depends only on the combinatorics of Δ. Given a combinatorial operation on Δ which produces another simplicial complex, it is natural to ask how the Stanley–Reisner ring of the new complex is related to that of Δ. Stellar subdivision, which is one of the simplest ways to subdivide a simplicial complex, is such an operation. It has been used successfully, for instance, to give a method for transforming the boundary of a polytope into that of any other polytope of the same dimension by operations which preserve interesting invariants [14], to construct polytopes whose f-vectors, or flag f-vectors, span a certain 'Euler' or 'Dehn-Sommerville' space [15, Chapter 9] [4] and to construct simplicial polytopes with prescribed face lattices [19, 28].

On a different tone, unprojection theory aims to analyze and construct commutative rings in terms of simpler ones. The first kind of unprojection which appeared in the literature is that of type Kustin–Miller, studied originally by Kustin and Miller [16] and later by Reid and the second author [24, 26]. Starting from a codimension one ideal I of a Gorenstein ring R such that the quotient R/I is Gorenstein, Kustin–Miller unprojection uses the information contained in $\mathrm{Hom}_R(I, R)$ to construct a new Gorenstein

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ring S which is 'birational' to R and corresponds to the 'contraction' of $V(I) \subset \operatorname{Spec} R$. It has been used in the classification of Tor algebras in Gorenstein codimension 4 [17], in the birational geometry of Fano 3-folds [11, 12], in the study of Mori flips [7], in the construction of algebraic surfaces of general type [22], in the construction of weighted K3 surfaces and Fano 3-folds [1], and in the construction of Calabi–Yau 3-folds of high codimension [5, 23]. A general discussion of unprojection theory and its applications is contained in [27], while a precise general definition of unprojection is proposed in [25].

The main objective of this paper is to show that the Stanley–Reisner rings of stellar subdivisions of a non-acyclic Gorenstein simplicial complex Δ can be constructed from the Stanley–Reisner ring of Δ by unprojections of type Kustin–Miller. As an application, we inductively calculate the minimal graded free resolutions of the Stanley–Reisner rings of the boundary simplicial complexes of stacked polytopes.

To state our main result, we need to introduce some notation and terminology (see Section 2 for more details). We denote by $k[\Delta]$ the Stanley–Reisner ring of a simplicial complex Δ with coefficients in a fixed field k. Recall that Δ is said to be Gorenstein* over k if $k[\Delta]$ is Gorenstein and Δ is non-acyclic over k. Given a face σ of Δ, we denote by Δ_σ the stellar subdivision of Δ on σ, by x_σ the square-free monomial in $k[\Delta]$ with support σ and by J_σ the annihilator of the principal ideal of $k[\Delta]$ generated by x_σ. Recall also from [26, Definition 1.2] that if $I = (f_1, \dots, f_r) \subset R$ is a homogeneous codimension 1 ideal of a graded Gorenstein ring R such that the quotient R/I is Gorenstein, then there exists $\phi \in \operatorname{Hom}_R(I, R)$ such that ϕ together with the inclusion $I \hookrightarrow R$ generate $\operatorname{Hom}_R(I, R)$ as an R-module. The Kustin–Miller unprojection ring of the pair $I \subset R$ is defined as the quotient of $R[y]$ by the ideal generated by the elements $yf_i - \phi(f_i)$, where y is a new variable.

Theorem 1.1. *Suppose that Δ is a Gorenstein* simplicial complex and that $\sigma \in \Delta$ is a face of dimension $d-1$ for some $d \geq 2$. Let z be a new variable of degree $d-1$ and set $M = \operatorname{Hom}_{k[\Delta][z]}((J_\sigma, z), k[\Delta][z])$.*

(a) *M is generated as a $k[\Delta][z]$-module by the elements i and ϕ_σ, where $i\colon (J_\sigma, z) \to k[\Delta][z]$ is the natural inclusion morphism, and ϕ_σ is uniquely specified by $\phi_\sigma(z) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$.*

(b) *Denote by S the Kustin–Miller unprojection ring of the pair $(J_\sigma, z) \subset k[\Delta][z]$. Then z is a S-regular element and $k[\Delta_\sigma]$ is isomorphic to $S/(z)$ as a k-algebra.*

An example demonstrating Theorem 1.1 is the following. Assume Δ is the boundary simplicial complex of the 2-simplex and σ is a facet of Δ. In coordinates, $k[\Delta] = k[x_1, x_2, x_3]/(x_1x_2x_3)$, $\sigma = \{1, 2\}$ and $J_\sigma = 0 : (x_1x_2) = (x_3)$. Then

$$S = \frac{k[x_1, \dots, x_4, z]}{(x_4z - x_1x_2, x_4x_3)},$$

where x_4 denotes the new unprojection variable. Notice that when $z = 0$, $S|_{z=0}$ is isomorphic to $k[\Delta_\sigma]$, while when $a \in k^*$, $S|_{z=a}$ is isomorphic (as ungraded k-algebra) to $k[\Delta]$. A toric face ring interpretation of S is discussed in Example 4.2.

The paper is organised as follows: Theorem 1.1 is proved in Section 3. Section 2 includes some definitions and background related to the concepts which appear in Theorem 1.1. Section 4 contains an interpretation of Theorem 1.1 using the theory of toric face rings. In Section 5, we apply Theorem 1.1 to inductively calculate the minimal graded free resolutions of the Stanley–Reisner rings of the boundary simplicial complexes of stacked polytopes. The graded Betti numbers of these rings were originally calculated in [31]. When the parameter value d is not 3, our methods allow us to obtain these Betti numbers without using Hochster's formula or Alexander duality. We conclude in Section 6 with some remarks and directions for future research.

We believe that the applications of unprojection theory to Stanley–Reisner rings are not limited to the case of stellar subdivisions, and in the paper [6] we use unprojection techniques for an inductive treatment of Stanley–Reisner rings associated to cyclic polytopes.

2. Preliminaries

Let m be a positive integer and set $E = \{1, 2, \dots, m\}$. An (abstract) *simplicial complex* on the vertex set E is a collection Δ of subsets of E such that (i) all singletons $\{i\}$ with $i \in E$ belong to Δ and (ii) $\sigma \subset \tau \in \Delta$ implies $\sigma \in \Delta$. The elements of Δ are called *faces* and those maximal with respect to inclusion are called *facets*. The dimension of a face σ is defined as one less than the cardinality of σ. The dimension of Δ is the maximum dimension of a face. The complex Δ is called *pure* if all facets of Δ have the same dimension. Any abstract simplicial complex Δ has a geometric realization, which is unique up to linear homeomorphism. When we refer to a topological property of Δ, we mean the corresponding property of the geometric realization of Δ.

For any subset ρ of E, we denote by x_ρ the square-free monomial in the polynomial ring $k[x_1, \dots, x_m]$ with support ρ. The ideal I_Δ of $k[x_1, \dots, x_m]$ which is generated by the square-free monomials x_ρ with $\rho \notin \Delta$ is called the *Stanley-Reisner ideal* of Δ. The *face ring*, or *Stanley-Reisner ring*, of Δ over k, denoted $k[\Delta]$, is defined as the quotient ring of $k[x_1, \dots, x_m]$ by the ideal I_Δ. Given a face σ of Δ of dimension at least 1, the *stellar subdivision* of Δ on σ is the simplicial complex Δ_σ on the vertex set $E \cup \{m+1\}$ obtained from Δ by removing all faces containing σ and adding all sets of the form $\tau \cup \{m+1\}$, where $\tau \in \Delta$ does not contain σ and $\tau \cup \sigma \in \Delta$. The complex Δ_σ is homeomorphic to Δ. We denote by J_σ the ideal $(0 : (x_\sigma))$ of $k[\Delta]$, in other words

$$J_\sigma = \{y \in k[\Delta] \colon y x_\sigma = 0\}.$$

The complex Δ is said to be Gorenstein* (over k) if $k[\Delta]$ is a Gorenstein ring and Δ is non-acyclic (over k). It is known [29, Section II.5] that Δ is Gorenstein* if and only if for any $\sigma \in \Delta$ (including the empty face) we have

$$\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma), k) \;\cong\; \begin{cases} k, & \text{if } i = \dim(\mathrm{lk}_\Delta(\sigma)) \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mathrm{lk}_\Delta(\sigma) = \{\tau \smallsetminus \sigma : \tau \in \Delta,\, \sigma \subset \tau\}$ is the link of σ in Δ and $\widetilde{H}_*(\mathrm{lk}_\Delta(\sigma), k)$ denotes simplicial homology of $\mathrm{lk}_\Delta(\sigma)$ with coefficients in the field k. By [9, Corollary 5.1.5], any Gorenstein* complex Δ is pure. It follows from (1) that the Gorenstein* property is inherited by links. In particular, any codimension 1 face of Δ is contained in exactly 2 facets of Δ. The class of Gorenstein* complexes includes all triangulations of spheres.

Assume R is a polynomial ring over a field k with the degrees of all variables positive, and M is a finitely generated graded R-module. Let

$$0 \to F_g \to F_{g-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

be the minimal graded free resolution of M as R-module. Write

$$F_i = \oplus_j R(-j)^{b_{ij}},$$

then b_{ij} is called the *ij-th graded Betti number* of M, and we also denote it by $b_{ij}(M)$. For more details about free resolutions and Betti numbers see, for example, [13, Sections 19, 20].

Assume R is a ring. An element $r \in R$ will be called *R-regular* if the multiplication by r map $R \to R, u \mapsto ru$ is injective. A sequence $r_1, \ldots, r_n$ of elements of R will be called a *regular R-sequence* if r_1 is R-regular, and, for $2 \le i \le n$, we have that r_i is $R/(r_1, \ldots, r_{i-1})$-regular.

3. Proof of Theorem 1.1

In this section, Δ denotes an $(n-1)$-dimensional simplicial complex on the vertex set $\{1, 2, \ldots, m\}$.

Remark 3.1. We will use the fact that $k[\Delta]$ has no nonzero nilpotent elements and that if I_1, I_2 are monomial ideals of $k[\Delta]$, then so is the ideal quotient

$$(I_1 : I_2) \;=\; \{y \in k[\Delta] \colon yI_2 \subset I_1\}.$$

Remark 3.2. Assume that Δ is Gorenstein*. If e is a vertex of Δ and $\sigma \in \Delta$ is a face that does not contain e, then there exists a facet of Δ that contains σ but not e. Indeed, let τ_1 be a facet of Δ containing σ. If τ_1 contains e, then there exists a facet τ_2 distinct from τ_1 containing $\tau_1 \setminus \{e\}$. This facet contains σ and does not contain e.

Proposition 3.3. *Let Δ be a Gorenstein* simplicial complex on the vertex set $\{1, 2, \ldots, m\}$ and let σ be a face of Δ of dimension at least 1. The ideal J_σ is a codimension 0 ideal of $k[\Delta]$ and the quotient $k[\Delta]/J_\sigma$ is Gorenstein. Moreover,*

$$(0 : J_\sigma) = (x_\sigma).$$

Proof. The first claim is well-known, cf. [13, Theorem 21.23], and the second follows from the observation that $k[\Delta]/J_\sigma = k[x_1, \ldots, x_m]/I$, where I is the Stanley-Reisner ideal of $\mathrm{lk}_\Delta(\sigma)$, and the fact that $\mathrm{lk}_\Delta(\sigma)$ is also Gorenstein*.

We now prove that $(0 : J_\sigma) = (x_\sigma)$. It is clear that $(x_\sigma) \subset (0 : J_\sigma)$. Since $(0 : J_\sigma)$ is a monomial ideal (Remark 3.1), it suffices to show that for any nonzero monomial $u \in (0 : J_\sigma)$ we have $u \in (x_\sigma)$. Let $\rho \in \Delta$ be the support of u. By the way of contradiction, suppose that u is not in (x_σ), so we may choose $i \in (\sigma \setminus \rho)$. By Remark 3.2, there exists a facet τ of Δ which does not contain i and contains ρ. Since i is not in τ and τ is a facet, we have $x_i x_\tau = 0$ in $k[\Delta]$ and hence $x_\tau \in J_\sigma$. This fact and the assumption $u \in (0 : J_\sigma)$ imply that $x_\tau u = 0$ in $k[\Delta]$. Since each variable which appears in u also appears in x_τ, we conclude that x_τ is a nonzero nilpotent element of $k[\Delta]$. This contradicts Remark 3.1 and completes the proof of the proposition. □

Remark 3.4. The conclusion of Proposition 3.3 is not true under the weaker hypothesis that $k[\Delta]$ is Gorenstein. For a counterexample consider

$$\Delta = \{\{1, 2\}, \{1, 3\}, \{1\}, \{2\}, \{3\}\}$$

and $\sigma = \{1, 2\}$. We have $k[\Delta] = k[x_1, x_2, x_3]/(x_2 x_3)$, $J_\sigma = (0 : x_1 x_2) = (x_3)$, but $(0 : J_\sigma) = (x_2)$. We believe that this is also a counterexample to the second claim of Part a) of [13, Theorem 21.23], this is the reason we did not use this claim in the proof of Proposition 3.3.

Let $\sigma \in \Delta$ be a face of dimension $d - 1$ for some $d \geq 2$. We recall that the stellar subdivision Δ_σ of Δ on σ is a simplicial complex on the vertex set $\{1, 2, \ldots, m + 1\}$. We will use the (easy) fact that

$$k[\Delta_\sigma] \cong \frac{k[x_1, \ldots, x_{m+1}]}{(I_\Delta, x_\sigma, x_{m+1}u_1, \ldots, x_{m+1}u_r)}, \tag{2}$$

where $\{u_1, \ldots, u_r\}$ is a generating set of monomials for the ideal J_σ of $k[\Delta]$.

Proof of Theorem 1.1. Clearly there exists a unique element ϕ_σ of M satisfying $\phi_\sigma(z) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$. Given $f \in M$, we write $f(z) = w_1 z + w_2$ with $w_1 \in k[\Delta][z]$ and $w_2 \in k[\Delta]$ and set $g = f - w_1 i \in M$, so that $g(z) = w_2$. For $u \in J_\sigma$ we have

$$zg(u) = g(zu) = ug(z) = uw_2 \in k[\Delta].$$

Hence $g(u) = 0$ for all $u \in J_\sigma$, which implies $w_2 \in (0 : J_\sigma)$. By Proposition 3.3 we have $(0 : J_\sigma) = (x_\sigma)$. As a consequence, there exist $w \in k[\Delta]$ such that $w_2 = w x_\sigma$ and hence $g = w\phi_\sigma$. This proves part (a) of the theorem.

By Proposition 3.3, the ring $k[\Delta]/J_\sigma$ is Gorenstein of the same dimension as $k[\Delta]$. Therefore (J_σ, z) is a codimension 1 homogeneous ideal of the graded Gorenstein ring $k[\Delta][z]$, so the general theory of [26] applies. Using part (a) we get

$$S \cong \frac{k[x_1, \ldots, x_{m+1}, z]}{(I_\Delta, x_{m+1}z - x_\sigma, x_{m+1}u_1, \ldots, x_{m+1}u_r)},$$

where the new variable x_{m+1} has degree equal to 1. It follows from (2) that $S/(z) \cong k[\Delta_\sigma]$. By [26, Theorem 1.5], S is Gorenstein of dimension equal to the dimension of $k[\Delta][z]$. As a consequence $\dim S/(z) = \dim S - 1$ and therefore z is an S-regular element. This completes the proof of the theorem. □

4. Toric face ring interpretation

Is it clear that Theorem 1.1 is equivalent to the following theorem.

Theorem 4.1. *Suppose that Δ is a Gorenstein* simplicial complex and that $\sigma \in \Delta$ is a face of dimension $d-1$ for some $d \geq 2$. Let $z_1, \dots, z_{d-1}$ be $d-1$ new variables of degree 1 and set $M_1 = \operatorname{Hom}_{k[\Delta][z_1,\dots,z_{d-1}]}((J_\sigma, z_1 z_2 \cdots z_{d-1}), k[\Delta][z_1, \dots, z_{d-1}])$.*

(a) *M_1 is generated as a $k[\Delta][z_1, \dots, z_{d-1}]$-module by the elements i and ϕ_σ, where $i \colon (J_\sigma, z_1 z_2 \cdots z_{d-1}) \to k[\Delta][z_1, \dots, z_{d-1}]$ is the natural inclusion morphism, and ϕ_σ is uniquely specified by $\phi_\sigma(z_1 z_2 \cdots z_{d-1}) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$.*

(b) *Denote by S_1 the Kustin–Miller unprojection ring of the pair $(J_\sigma, z_1 z_2 \cdots z_{d-1}) \subset k[\Delta][z_1, \dots, z_{d-1}]$. Then $z_1, z_2, \dots, z_{d-1}$ is an S_1-regular sequence, and $k[\Delta_\sigma]$ is isomorphic to $S_1/(z_1, z_2, \dots, z_{d-1})$ as a k-algebra.*

We remark that, unlike in Theorem 1.1, in Theorem 4.1 all variables have degree 1 which is the usual grading in the theory of Stanley–Reisner rings. Compare also [6, Section 4], where a similar product $z_1 z_2$ appears in a natural way when relating unprojection and cyclic polytopes.

Consider the Kustin–Miller unprojection ring

$$S_1 = \frac{k[x_1, \dots, x_{m+1}, z_1, \dots, z_{d-1}]}{(I_\Delta, x_{m+1} z_1 \cdots z_{d-1} - x_\sigma, x_{m+1} u_1, \dots, x_{m+1} u_r)}$$

appearing in Theorem 4.1, where as in Section 3 $\{u_1, \dots, u_r\}$ denotes a generating set of monomials for the ideal $J_\sigma = (0 : x_\sigma)$ of $k[\Delta]$. We will now give a combinatorial interpretation of S_1 using the notion of toric face rings as defined by Stanley in [30, p. 202], compare also [8, Section 4] and [10]. Let M be a free $\mathbb{Z}$-module of rank $m+d-1$, and consider the $\mathbb{R}$-vector space $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. We will define a (finite, pointed) rational polyhedral fan $\mathcal{F}$ in $M_\mathbb{R}$, such that S_1 is isomorphic to the toric face ring $k[\mathcal{F}]$. For simplicity of notation we assume in the following that $\sigma = \{1, 2, \dots, d\}$.

Denote by $e_{x,1}, \dots, e_{x,m}, e_{z,1}, \dots, e_{z,d-1}$ a fixed $\mathbb{Z}$-basis of M, and set

$$e_a = (e_{x,1} + \cdots + e_{x,d}) - (e_{z,1} + \cdots + e_{z,d-1}) \in M.$$

Assume $\tau = \{a_1, \dots, a_p\}$ is a face of Δ. If σ is not a face of τ we set c_τ to be the cone in $M_\mathbb{R}$ spanned by the basis vectors

$$e_{x,a_1}, \dots, e_{x,a_p}, e_{z,1}, \dots, e_{z,d-1},$$

while if σ is a face of τ we set c_τ to be the cone in $M_{\mathbb{R}}$ spanned by the (non-affinely independent) vectors

$$e_{x,a_1}, \ldots, e_{x,a_p}, e_{z,1}, \ldots, e_{z,d-1}, e_a.$$

It is easy to see that the collection of cones $\{c_\tau \mid \tau \text{ face of } \Delta\}$ together with their faces form a fan $\mathcal{F}$ in $M_{\mathbb{R}}$ and that the toric face ring $k[\mathcal{F}]$ is isomorphic as a k-algebra to S_1.

Example 4.2. *Consider the example given after the statement of Theorem 1.1. That is, let Δ be the boundary of a triangle with vertices corresponding to the variables x_1, x_2, x_3, and denote by Δ_σ the stellar subdivision of Δ with respect to the face x_1x_2. We embed the fan $\mathcal{F}$ into $\mathbb{R}^3$ by assigning to the variables x_1, x_2, x_3, x_4 the rays generated by $(1,0,0)$, $(0,1,0)$, $(-1,-1,-1)$, $(0,0,1) \in \mathbb{Z}^3$, i.e., those of the standard fan of $\mathbb{P}^3$ as a toric variety. Then the ray associated to z is generated by $(1,1,-1)$. The right hand side of Figure 1 visualizes the Kustin-Miller unprojection ring $S \cong k[\mathcal{F}]$ via representing each cone of the embedded fan $\mathcal{F}$ by a polytope spanning it. There are 3 polytopes of maximal dimension, spanned by $\{x_1, x_3, z\}, \{x_2, x_3, z\}$ and $\{x_1, x_4, x_2, z\}$. Notice that subdividing the cone corresponding to x_1, x_4, x_2, z into x_1, x_4, z and x_4, x_2, z amounts to passing from S to the polynomial ring in the variable z over $k[\Delta_\sigma]$.*

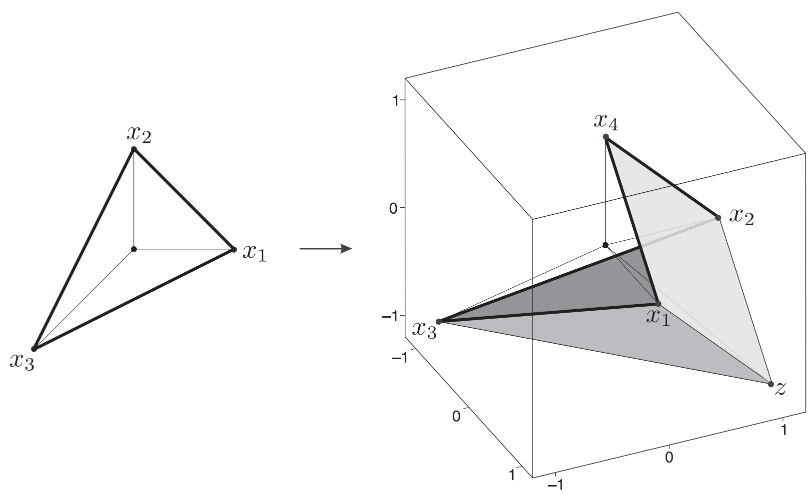

FIGURE 1. Unprojection via toric face rings

5. Application to stacked polytopes

5.1. **The Kustin–Miller complex construction.** The following construction, which is due to Kustin and Miller [16], will be important in the applications to stacked polytopes contained in Subsection 5.2.

Assume R is a polynomial ring over a field with the degrees of all variables positive, and $I \subset J \subset R$ are two homogeneous ideals of R such that both quotient rings R/I and R/J are Gorenstein and $\dim R/J = \dim R/I - 1$. We define $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$, compare [9, Proposition 3.6.11], and assume that $k_1 > k_2$. Moreover, let

$$0 \to A_g \to A_{g-1} \to \ldots \to A_1 \to A_0 \to R/J \to 0$$

and

$$0 \to B_{g-1} \to \ldots \to B_1 \to B_0 \to R/I \to 0$$

be the minimal graded free resolutions of R/J and R/I respectively as R-modules. Denote by $S = R[T]/Q$ the Kustin–Miller unprojection ring of the pair $J \subset R/I$, where T is a new variable of degree $k_1 - k_2$. Kustin and Miller constructed in [16] a graded free resolution of S as $R[T]$-module of the form

$$0 \to F_g \to F_{g-1} \to \ldots \to F_1 \to F_0 \to S \to 0,$$

where, when $g \geq 3$,

$$\begin{aligned} &F_0 = B_0', \qquad F_1 = B_1' \oplus A_1'(k_2 - k_1), \\ &F_i = B_i' \oplus A_i'(k_2 - k_1) \oplus B_{i-1}'(k_2 - k_1), \qquad \text{for } 2 \leq i \leq g-2, \\ &F_{g-1} = A_{g-1}'(k_2 - k_1) \oplus B_{g-2}'(k_2 - k_1), \qquad F_g = B_{g-1}'(k_2 - k_1), \end{aligned}$$

cf. [16, p. 307, Equation (3)]. When $g = 2$ we have

$$F_0 = B_0', \quad F_1 = A_1'(k_2 - k_1), \quad F_2 = B_1'(k_2 - k_1).$$

In the above expressions, for an R-module M we denoted by M' the $R[T]$-module $M \otimes_R R[T]$. This resolution is, in general, not minimal, see Example 5.2 below. However, in the case of stacked and cyclic polytopes it is minimal, see Theorem 5.3 and [6]. We call the complex consisting of the F_i the *Kustin–Miller complex construction.*

Example 5.1. Assume $k_2 - k_1 = -1$, and that the 2 complexes are

$$0 \to A_4 \to A_3 \to A_2 \to A_1 \to A_0 \to 0$$

and

$$0 \to B_3 \to B_2 \to B_1 \to B_0 \to 0$$

Then, the Kustin–Miller complex construction is of the form

$$0 \to B_3'(-1) \to B_2'(-1) \oplus A_3'(-1) \to B_1'(-1) \oplus A_2'(-1) \oplus B_2' \to A_1'(-1) \oplus B_1' \to B_0' \to 0$$

Example 5.2. Let Δ be the simplicial complex with Stanley–Reisner ideal $(x_1x_2x_3, x_4x_5)$, Δ is just the stellar subdivision of a facet of the boundary complex of the 3-simplex. Then $\sigma = \{1, 2\}$ is a face of Δ, and one can easily check that the Kustin–Miller complex construction gives a graded resolution of $k[\Delta_\sigma]$ which is not minimal, the reason being that the Stanley-Reisner ideal of Δ_σ needs 3 generators and not 5.

5.2. **The minimal resolution for stacked polytopes.** Assume $d \geq 2$ is a fixed integer. Recall from [31, p. 448], that starting from a d-simplex one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex d-polytope with m vertices, called a *stacked polytope* $P_d(m)$. We denote by $\Delta P_d(m)$ the boundary simplicial complex of the simplicial polytope $P_d(m)$. By definition, $\Delta P_d(m)$ has as elements the empty set and the sets of vertices of the proper faces of $P_d(m)$, cf. [9, Corollary 5.2.7]. There is a slight abuse of notation here, since the combinatorial type of $\Delta P_d(m)$ does not depend only on d and m but also on the specific choices of the sequence of facets we used when building the shallow pyramids. The graded Betti numbers b_{ij} of the Stanley-Reisner ring $k[\Delta P_d(m)]$ have been calculated in [31, Theorem 1.1], and it turns out that they only depend on d and m.

It is clear that, for $d < m$, the simplicial complex $\Delta P_d(m+1)$ can be considered as the stellar subdivision of the boundary simplicial complex $\Delta P_d(m)$ of a stacked polytope $P_d(m)$ with respect to a facet σ of $\Delta P_d(m)$. Since σ is a facet, the ideal (J_σ, z) is generated by the regular sequence x_ρ, z, where ρ takes values in the set of vertices of $\Delta P_d(m)$ which are not vertices of σ. Hence, the minimal graded free resolution of (J_σ, z) is a Koszul complex. Combining Theorem 1.1 with the Kustin–Miller complex construction described in Subsection 5.1 we can get, starting with the Koszul complex and the minimal graded free resolution of $k[\Delta P_d(m)]$, a graded free resolution of $k[\Delta P_d(m+1)]$. The following theorem, whose proof will be given in Subsection 5.3, states that in this way we get a minimal graded free resolution of $k[\Delta P_d(m+1)]$. We remark that, when $d=2$ or $d \geq 4$, we do not use in the proof of the theorem the calculation of the graded Betti numbers of $k[\Delta P_d(m)]$ obtained in [31], and, moreover, we recover these numbers in Proposition 5.7.

Theorem 5.3. *Assume $d \geq 2$ and $d+1 < m$. The resolution of $k[\Delta P_d(m+1)]$, obtained using the Kustin–Miller complex construction starting from the minimal graded free resolution of $k[\Delta P_d(m)]$ and the Koszul complex resolving (J_σ, z) is minimal.*

5.3. **Proof of Theorem 5.3.** We need the following combinatorial definition. Assume $d \geq 2$ and $d < m$. For $1 \leq i \leq m-d-1$ we define

$$\theta(d,m,i) = i\binom{m-d}{i+1},$$

compare [31, p. 448]. Moreover we set $\theta(d,m,0) = \theta(d,m,m-d) = 0$.

Lemma 5.4. (Compare [31, p. 451]). *Assume $1 \leq i \leq m-d$. Then*

$$\theta(d,m+1,i) = \theta(d,m,i) + \binom{m-d}{i} + \theta(d,m,i-1).$$

(By our conventions, for $i=1$ the equality becomes $\theta(d,m+1,1) = \theta(d,m,1) + (m-d)$, while for $i = m-d$ it becomes $\theta(d,m+1,m-d) = \theta(d,m,m-d-1)+1$).

Proof. Assume first $2 \leq i \leq m-d-1$. Using the Pascal triangle identity $\binom{m}{d} = \binom{m-1}{d} + \binom{m-1}{d-1}$ we have

$$\begin{aligned}\theta(d, m+1, i) &= i\binom{m+1-d}{i+1} = i(\binom{m-d}{i+1} + \binom{m-d}{i}) \\ &= i\binom{m-d}{i+1} + \binom{m-d}{i} + (i-1)\binom{m-d}{i} \\ &= \theta(d,m,i) + \binom{m-d}{i} + \theta(d,m,i-1).\end{aligned}$$

The special cases $i = 1$ and $i = m-d$ are proven by the same argument. □

The following proposition is well-known.

Proposition 5.5. ([9, Proposition 1.1.5]). *Assume $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k with the degrees of all variables positive, and $I \subset R$ a homogeneous ideal. Moreover, assume that x_n is R/I-regular. Denote by cF the minimal graded free resolution of R/I as R-module. We then have that $cF \otimes_R R/(x_n)$ is the minimal graded free resolution of $R/(I, x_n)$ as $k[x_1, \ldots, x_{n-1}]$-module, where we used the natural isomorphisms $R \otimes_R R/(x_n) \cong R/(x_n) \cong k[x_1, \ldots, x_{n-1}]$.*

The proof of the following proposition is an immediate corollary of the construction of the Koszul complex in [9, Section 1.6].

Proposition 5.6. *Assume $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k with the degrees of all variables positive, $p \leq n$ a fixed integer, and $g_1, \ldots, g_p$, an R-regular sequence consisting of homogeneous elements of R, with $\deg g_i = 1$, for $1 \leq i \leq p-1$, and $\deg g_p = q \geq 1$. Then, the minimal resolution of $R/(g_1, \ldots, g_p)$ is of the form*

$$0 \to F_p \to F_{p-1} \ldots \to F_1 \to F_0,$$

with $F_0 \cong R$, $F_p \cong R(-p-q+1)$, and

$$F_i \cong R(-i)^{b_i} \oplus R(-q-i+1)^{b_{p-i}}$$

for $1 \leq i \leq p-1$, where $b_i = \binom{p-1}{i}$.

Theorem 5.3 will follow from the following more precise proposition. Notice that as we have already mentioned the statements about the graded Betti numbers of $k[\Delta P_d(m)]$ have been proven before in [31]. For $d \neq 3$ we do not use in our proof the results of [31].

Proposition 5.7. *Assume $d \geq 2$ and $d+1 < m$. Set $b_{ij} = b_{ij}(k[\Delta P_d(m)])$. Then the statement of Theorem 5.3 is true for (d, m). Moreover, we have that if $d = 2$ then $b_{ij} = 1$ for $(i, j) \in \{(0, 0), (m-d, m)\}$,*

$$b_{i,i+1} = \theta(d, m, i) + \theta(d, m, m-d-i),$$

for $1 \leq i \leq m-d-1$, and $b_{ij} = 0$ otherwise. If $d \geq 3$, we have $b_{ij} = 1$ for $(i, j) \in \{(0, 0), (m-d, m)\}$,

$$b_{i,i+1} = \theta(d, m, i), \quad b_{i,d+i-1} = \theta(d, m, m-d-i),$$

for $1 \leq i \leq m-d-1$*, and* $b_{ij} = 0$ *otherwise.*

Proof. We fix $d \geq 2$ and use induction on m. If $d \geq 2$ and $m = d+2$ then $k[\Delta P_d(m)]$ is a type $(2, d)$ codimension 2 complete intersection and everything is clear.

Assume $d \neq 3$, and that Proposition 5.7 is true for (d, m). By Theorem 1.1, the extension ring S of $k[\Delta P_d(m+1)]$ is the Kustin–Miller unprojection ring of the pair $(J_\sigma, z) \subset k[\Delta P_d(m)][z]$. As we noticed above, the ideal (J_σ, z) is generated by a regular sequence, so the Koszul complex described in Proposition 5.6 is the minimal resolution of $k[\Delta P_d(m)][z]/(J_\sigma, z)$. Combining Proposition 5.5 and the discussion of Subsection 5.1, starting from the Koszul complex and the minimal graded free resolution of $k[\Delta P_d(m)]$, the Kustin–Miller complex construction gives a graded free resolution of $k[\Delta P_d(m+1)]$. Using Lemma 5.4 this complex has the conjectured graded Betti numbers, and since there are no degree 0 morphisms it is necessarily minimal.

When $d = 3$ the above arguments work except for the minimality argument, since there are degree 0 morphisms. But comparing the graded Betti number of the Kustin–Miller complex construction with the graded Betti numbers of $k[\Delta P_d(m+1)]$ calculated in [31] we again obtain the minimality of the Kustin–Miller complex construction. □

6. Remarks and open questions

In the following we use the notation introduced in Section 1 and Theorem 1.1.

Remark 6.1. It follows from Theorem 1.1 that S is a 1-parameter deformation ring of $k[\Delta_\sigma]$, compare [13, Exerc. 18.18]. The fact that such a deformation ring of $k[\Delta_\sigma]$ exists is a special case of more general results due to Altmann and Christophersen [2, 3].

Remark 6.2. Using the Kustin–Miller complex construction described in Subsection 5.1, we can construct a graded free resolution of S, therefore using Proposition 5.5 also of $k[\Delta_\sigma]$, starting from graded free resolutions of $k[\Delta]$ and $k[\Delta]/J_\sigma$. In particular, it follows that

$$F(k[\Delta_\sigma], t) = F(k[\Delta], t) + (t + t^2 + \cdots + t^{d-1})\, F(k[\Delta]/J_\sigma, t),$$

where $F(R, t)$ stands for the Hilbert series of R and $d-1$ is the dimension of the face σ. This equality can be rewritten as

$$h(\Delta_\sigma, t) = h(\Delta, t) + (t + t^2 + \cdots + t^{d-1})\, h(\mathrm{lk}_\Delta(\sigma), t), \tag{3}$$

where $h(\Gamma, t)$ stands for the h-polynomial [29, Section II.2] of the simplicial complex Γ. It is not hard to see that (3) holds for any pure simplicial complex Δ. Indeed, one can check directly that (3) is equivalent to the formula

$$f_j(\Delta_\sigma) = f_j(\Delta) - f_{j-d}(\mathrm{lk}_\Delta(\sigma)) + \sum_{i \geq 0} \binom{d}{j-1} f_{i-1}(\mathrm{lk}_\Delta(\sigma)),$$

where $f_j(\Gamma)$ denotes the number of j-dimensional faces of a complex Γ. That formula follows from the definition of Δ_σ.

Remark 6.3. In [22], Neves and the second author introduced the $\binom{n}{2}$ Pfaffians format, starting from a certain hypersurface ideal. We give a monomial interpretation of the construction. Start with the boundary simplicial complex Δ of the $(n-1)$-simplex. Denote by Δ_1 the simplicial complex obtained by the stellar subdivisions of all facets of Δ. It is easy to check that the Stanley–Reisner ideal of Δ_1 is equal to $\widetilde{I_n}$, where $\widetilde{I_n}$ denotes the ideal obtained by substituting $z_i = 0$, for $1 \leq i \leq n$, and $r_{d_1,\dots,d_n} = 1$, for $(d_1, \dots, d_n) \in \{0,1\}^n$, to the ideal I_n defined in [22, Definition 2.2].

Similarly, in [23, Section 4.3], Neves and the second author constructed a codimension 11 Gorenstein ideal starting from a certain codimension 2 complete intersection ideal. The monomial interpretation of the construction is as follows. Denote by Δ the simplicial complex which is the join [9, p. 221] of 2 copies of the boundary simplicial complex of the 2-simplex. Δ has Stanley–Reisner ideal equal to $(x_{11}x_{12}x_{13}, x_{21}x_{22}x_{23})$ and exactly 9 facets. Denote by Δ_1 the simplicial complex obtained by the stellar subdivisions of Δ on these 9 facets. Using the notations of [23, Section 2], denote by $I_{\mathcal{L}}$ the kernel of the surjection $R[y_u \mid u \in \mathcal{L}] \to R_{\mathcal{L}}$. It is easy to check that the Stanley–Reisner ideal of Δ_1 is equal to $\widetilde{I_{\mathcal{L}}}$, where $\widetilde{I_{\mathcal{L}}}$ denotes the ideal obtained by substituting $x_{3i} = 0$, for $1 \leq i \leq 3$, to $I_{\mathcal{L}}$.

Remark 6.4. Theorem 1.1 suggests that, given a Gorenstein* simplicial complex Δ, the essential information about the relationship between the Stanley–Reisner rings of Δ and a stellar subdivision Δ_σ of Δ is contained in the module $\mathrm{Hom}_{k[\Delta][z]}((J_\sigma, z), k[\Delta][z])$. We believe that for many other natural combinatorial operations on simplicial complexes, the key information about the relationship between the corresponding Stanley–Reisner rings is contained in suitable Hom modules over the original ring. We give below two examples, compare also [6] for the case of cyclic polytopes.

For the first example, consider the simplicial complex Δ with vertex set $\{1, 2, \dots, 6\}$ and facets

$$\{1,2,3\}, \{1,3,4\}, \{1,4,5\}, \{1,5,6\}, \{1,2,6\}, \{2,3,4\}, \{2,4,5\}, \{2,5,6\}$$

and the simplicial complex Δ' with vertex set $\{1, 2, \dots, 7\}$ and facets

$$\{1,2,3\},\ \{1,2,6\},\ \{1,3,4\},\ \{1,4,5\},\ \{1,5,6\},\ \{2,3,7\},\ \{2,6,7\},\ \{3,4,7\},\ \{4,5,7\},\ \{5,6,7\}.$$

The complex Δ is combinatorially isomorphic to the boundary simplicial complex of a 3-dimensional cyclic polytope with 6 vertices. It can be obtained from Δ' by 'sliding' (moving) vertex 7 to vertex 2. Set $R = k[\Delta][z]$ and $I_1 = (x_1, z) \subset R$, where $k[\Delta] = k[x_1, \dots, x_6]/I_\Delta$. It can be shown that $\mathrm{Hom}_R(I_1, R)$ is generated as an R-module by the inclusion morphism $I_1 \hookrightarrow R$ together with two elements ϕ_1, ϕ_2 defined by

$$\phi_1(x_1) = \phi_2(x_1) = 0, \qquad \phi_1(z) = x_2x_4,\ \phi_2(z) = x_2x_5.$$

Moreover, if we set

$$I_2 = I_\Delta + (x_1x_7, x_7z - \phi_1(z), x_1x_8, x_8z - \phi_2(z)) \subset k[x_1, \ldots, x_8, z],$$

then we have

$$k[\Delta'] \cong \frac{k[x_1, \ldots, x_8, z]}{(I_2, z, x_8 - x_7)}.$$

For an example in a different direction, consider the boundary simplicial complex Δ of the standard 3-simplex on the vertex set $\{1, 2, 3, 4\}$ and the simplicial complex Δ' with vertex set $\{1, 2, \ldots, 7\}$ and facets

$$\{1,2,5\},\ \{1,3,5\},\ \{1,3,6\},\ \{1,4,6\},\ \{1,4,7\},\ \{1,2,7\},\ \{2,5,7\},\ \{3,5,6\},\ \{4,6,7\},\ \{5,6,7\}.$$

We note that Δ' is a subdivision of Δ which cannot be obtained from Δ by a sequence of stellar subdivisions. We set $R = k[\Delta][z]$, where $k[\Delta] = k[x_1, x_2, x_3, x_4]/I_\Delta$ and, for $2 \leq i \leq 4$, $I_i = (x_1x_i, z) \subset R$. For $2 \leq i \leq 4$ we have that $\operatorname{Hom}_R(I_i, R)$ is generated as an R-module by the inclusion morphism $I_i \hookrightarrow R$ together with a single element ϕ_i such that $\phi_i(x_0x_i) = 0$ and $\phi_i(z) = x_ax_b$, with $\{i, a, b\} = \{2, 3, 4\}$. We denote by $K(R)$ the total ring of fractions of R, defined as the localization of R with respect to its multiplicative subset of regular elements. Consider the three elements $\tilde{s}_i = x_ax_b/z \in K(R)$ (indices as before) and denote by A the R-subalgebra of $K(R)$ generated by R and $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4$. It can be shown that $k[\Delta']$ is isomorphic to the quotient of the ring $A/(z)$ by its radical.

Remark 6.5. It is plausible that our ideas also generalize to non-Gorenstein simplicial complexes. To do this a more detailed study of non-Gorenstein unprojections would be necessary.

Remark 6.6. Combining the results of the present note with those of [18] we get a link between stellar subdivisions of non-acyclic Gorenstein simplicial complexes and linkage theory [20]. Is it possible to use this connection to define new combinatorial invariants of simplicial complexes?

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